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Lagrange Interpolation Polynomial for the Degrees of Non - Reducible Representations of p– Groups for Even Primes

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Abstract

Jelten N et al (2017) and (2019) in their work derived respectively schemes for the minimum and maximum number of the non - reducible (irreducible) representations of finite non- abelian 2 – groups. In this paper we derive using the Lagrange interpolation the Lagrange polynomials for the minimum and maximum non - reducible representations of finite p -groups for even p. The minimum numerical values we generated as part of our results were from the scheme $|C| \leq \frac{|G|+12}{4}$ as in theorem 1.18 of this paper while the maximum values generated as parr of our results were obtained from the scheme $|C| \leq \frac{5}{8}|G|$ as in theorem 1.19 of our paper. Here we express properties of abstract structures in numerical forms where we write |G| as card(G) and |C| as min rep(G) and max (G) for the minimum and maximum respectively. With this we derive entirely new results creating a relationship between the abstract part of pure mathematics and the numerical part of applied mathematics. Surprisingly our results agree with those of Jelten, N et al. The schemes we obtain have fantastic properties as they can be used to generate minimum and maximum non- reducible representations of other groups by inserting integral values. This is a characteristic of numerical analysis. In our work we employ the results of Jelten, N. et al (2017) and (2019) and Lagrange interpolation polynomial as the main tools to derive our results.

KEY WORDS: centre, abelian, group, Lagrange's polynomial, representation

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1. PRELIMINARIES ON GROUPS

Here we define some basic and fundamental concepts and theorems on groups.

Definition 1.1

A group G is a set closed for a binary operation in G satisfying the following axioms:

(i) The operation on the set G is associative. That is $a\,(bc) = (ab)c$ where a, b and c are elements of G. (ii) There exists an element $e \in$ G called the identity element in G such that: $ae = a = ea$.

(iii) The set G possesses an element $a^{-1} \in G$ called the inverse of a such that: $aa^{-1} = e = a^{-1}a$.

A group which possesses the property that for any pair of elements x, y in G we have $xy = yx$ is called an abelian group and non-abelian if $xy \neq yx$..

Definition 1.2

The number of elements in a group G written $|G|$ is called the cardinality of the group. If G is finite of order n we write $|G| = n$ otherwise, $|G| = \infty$ if G has infinite order. The least number n if it exists such that $a^n = 1$for a in G is called the order of element a and we write $o(a) = n$ and $o(a) = \infty$ if no such n exists.

Definition 1.3

A non-empty subset N of a group G is said to be a subgroup of G written $N \leq G$, if N is a group under the operation inherited from G. If $N \neq G$, then N is a proper subgroup of G..

Definition 1.4

Let G be a group and $H < G$. For $q \in G$ the subset Hq of G is called the right coset of H defined as $Hq = \{hq : h \in H\}$. Distinct right cosets of H in G form a partition of G. Left coset is similarly defined. The number of distinct right cosets of H in G is written as $|G{:}H|$ and is called the index of H in G. If G is finite so is H and G is partitioned into $|G{:}H|$ cosets each of order $|H|$. It follows that: $|G{:}H| = \frac{|G|}{|H|}$ The next theorem gives a relationship between the order of a group and the order of its subgroups.

Theorem 1.5

If a group G is finite and H is a subgroup of G then the order of H divides the order of G.

Proof

By definition 1.6 we have that the right cosets of H form a partition of G. Thus, each element of G belongs to at least one right coset of H in G and no element can belong to two distinct right cosets of H in G. Therefore, every element of G belongs to exactly one right coset of H. Moreover, each right coset of H in G contains $|H|$ elements. Therefore, if the number of right cosets of H in G is n, then $|G| = n|H|$. Hence the order of H divides the order of G.

Definition 1.6

If $g \in G$ is such that g generates the group G we say G is cyclic written as $G = < g >$. That is $G = \{ g^n : n \in Z\}$ if G is finite. In the infinite case, G is cyclic if the powers of g exhaust G. Here, every element $b \in G$ can be expressed as $b = g^m$ for some integer m.

Definition 1.7

Let a, $q \in G$. Then a is said to be the conjugate of q in G if there exist an element $g \in G$ such that $q = g^{-1} a g$. The set of all elements of G that are conjugate to a in G is called the conjugacy class of a in G which we denote by $C(a)$ and defined as: $C(a) = \{ g^{-1} a g | g \in G\}$. We note that $C(a)$ is a subgroup of G and by theorem 1.5 its order divides that of G.

Definition 1.8

The centre $Z(G)$ of a group G is the set of all elements z in G that commute with every element q in G.and is defined as $Z(G) = \{z \in G | zq = qz$, for all $q \in G\}$ The centralizer $C_G(q)$ of an element q in G is the set of all elements $g \in G$ that commute with q. This is defined as: $C_G(q) = \{g \in G : gq=qg$, for some $q \in G\}$. Joseph G. (2011) proves that the centre and centralizer are subgroups of G.

Remark 1.9

Note that H and G/H are subgroups of G. From theorem 1.5, $|H|$ and $|G:H|$ divide $|G|$. A consequence of this is that if H is a subgroup of G then, $|G| = |G:H||H|$.

The centralizer is a subgroup of G hence its order divides |G|. The index of $C_G(q)$ in G is the size of the conjugacy class C(q) of q in G. That is $|C(q)| = |G:C_G(q)|$. If $q \in Z(G)$ then $|C(q)| = 1$ and $q^{-1} gq = q$. So that $C_G(q) = G$.

As a consequence of theorem 1.5 the size of the conjugacy class is a factor of the order of G. If we choose a single representative element x_i from each conjugacy class, then $|G| = \sum|G:C_G(x_i)|$. This gives rise to an important theorem, the class equation which we state in Theorem 1.12

Next, we define one of the tools in this paper

Definition 1.10

A p - group is a group whose order is a power of a prime p. If a group G has order p^m where p is a prime and m is a positive integer, then G is a p - group. Consequently 2- groups are p – groups with p = 2.

From definition 1.7 we have a theorem on the relationship between the order of the conjugacy class of an element of G and its index in G as follows:

Theorem 1.11

Let G be a finite group and $q \in G$, then the conjugacy class C(q) of q in G is given by:

$|C(q)| = |G:C_G(q)| = \frac{|G|}{|C_G(q)|}$. The sum of the centralizers of all elements of G can be separated into sums of the centralizers of all the elements from each conjugacy class of G.

Theorem 1.12

Let G be a finite group then from remark 1.9 $|G| = \sum|G: C_G(q_i)|$ and from Theorem 1.11, we have $|G| = |Z(G)| + \sum|G: C_G(q_i)|$ which is called the class equation. $|G| = |Z(G)| + \sum|C(q_i)|$ is another form of class equation.

Theorems 1.5 and 1.12 lead to:

Proposition 1.13

If the order of a finite group G is a power of a prime p then G has a non-trivial centre. Equivalently the centre of a p - group contains more than one element.

C, Cody (2010) proved the next theorem.

Theorem 1.14

If G is a finite non abelian group, then the maximum possible order of the centre of G is ¼$|G|$. That is, $|Z(G)| \leq 1/4|G|$.

In the next proposition we relate the conjugacy class to the non - reducible representation of a finite group G as in C, Cody (2010)

Proposition 1.15

(i) The number of the non - reducible representations of any group G is equal to the number of conjugacy classes of G; (ii) Every non - reducible representation of a commutative group G over C, the set of complex numbers is one dimensional.

J, B. Naphtali and E, Apine (2015), proved the next theorem

Theorem 1.16

Given that a finite group G is of prime power order with centre Z(G), then we count the number of conjugacy classes from the centralizer as follows:

$$|C| \leq \frac{1}{4}(3|G| - |C_G(x)|)$$

From Jelten, et all (2017) the theorem that follows holds trivially.

Theorem 1.17

Let G be a finite p-group with centre |Z(G)|. Then the number of non - reducible representations of G is |G| if the group G is abelian with centre at its minimum given $G^{/} \leq G$.

Theorems 1.18 and 1.19 are respectively proved in Jelten elt 2017 and 2019 are major theorems of reference in this paper.

Theorem 1.18

Let G be a finite non abeian 2-group with centre |Z(G)| fixed at its minimum. Then the number of irreducible representations of G is given by $|C| \leq \frac{|G|+12}{4}$ given that $G^{/} \leq G$.

Theorem 1.19

For a finite non - abelian 2 – group with a maximum centre the maximum number of irreducible representations is at most $|C| \leq \frac{5}{8}|G|$.

2. ELEMENTARY DEFINITIONS FROM NUMERICAL ANALYSIS

Remark 2.1

Our focus in this paper is on the Lagrange's form of interpolating polynomial for a given data. It is known that the interpolating polynomial for a given data is unique

From Erwin Kreyszig (2002), we extract the following definitions and remarks

Definition 2.2

Interpolation means to find (approximated) values of a function f(x) for an x between different x – values $x_0, x_1, ..., x_n$ at which the values of f(x) are given. The values

$f_0 = f(x_0), f_1 = f(x_1), ..., f_n = f(x_n)$

may come from a mathematical function given by a formular or from an empirical function resulting from observation or experiment.

Definition 2.3

A standard idea in interpolation now is to find a polynomial pn(x) of degree n or less that assumes the given values, thus $p_n(x_0) = f_0, p_n(x_1) = f_1, ..., p_n(x_n) = f_n)$

We call this p_n an interpolation polynomial and $x_0, x_1, ..., x_n$, nodes. And if f(x) is a mathematical function, we call p_n an approximation of f or a polynomial approximation, because there are other kinds of approximation (which are not relevant to this paper). We us p_n to get (approximate) values of f for x's between x_0 and x_n (interpolation) or sometimes outside the interval (extrapolation)

Remark 2.4

Polynomials are convenient to work with because we can readily differentiate and integrate them again obtaining polynomials. Moreover, they approximate continuous functions with any desired accuracy. That leads to the next theorem called Weierstrass theorem

Theorem 2.5

For any continuous f(x) on an interval $J, a \leq x \leq b$ an error bound $\beta > 0$, there is a polynomial p_n (x) (of sufficiently high degree n) such that $|f(x) - p_n(x)| < \beta$ for all x in J.

This theorem is proved by Kreyszig, E (1989). Next, we prove the existence and uniqueness of p_n as in Erwin Kreyszig (2002)

Theorem 2.6

p_n satisfying the values in definition 2.3 for a given data exists and p_n is unique.

Proof

Indeed, if a polynomial q_n also satisfies

$q_n(x_0) = f_0, q_n(x_1) = f_1, ..., q_n(x_n) = f_n)$ then $p_n(x) - q_n(x) = 0$ at $x_0, x_1, ..., x_n$, but a polynomial $p_n - q_n$ of degree n or (less) with n + 1 roots must be identically zero, as we know from algebra, thus $p_n(x) \equiv q_n(x)$ for all x, which means uniqueness

Remark 2.7

Several methods are employed for finding p_n whci include Lagrange interpolation polynomial, newtons divided difference interpolation, inverse interpolation and the Taylor's series.

In this work, we employ the Lagrange interpolation polynomial to obtain our results

Lagrange interpolation 2.8

Given $(f_0, x_0), (f_1, x_1), \ldots, (f_n, x_n)$ with arbitrary spaced x_j.Lagrange has the idea of multiplying each f_j by a polynomial that is 1 at x_j and 0 at the other n nodes, and then to take the sum of these n + 1 polynomials to get the unique interpolation polynomial of degree n or less

Linear interpolation 2.8.1

Interpolation by straight line through (x_0, f_0), (x_j, f_j), gives rise to the linear lagrange polynomial p1 as $p_1 = L_0 f_0 + L_1 f_1$ with L_0 the linear polynomial that is 1 at x_0 and 0 at x_1 , similarly L_1 is 0 at x_0 and 1 at x_1.From the foregoing it's obvious that:

$L_0(x) = \frac{(x - x_1)}{(x_0 - x_1)}$, $L_1(x) = \frac{(x - x_0)}{(x_1 - x_0)}$, from this we obtain the linear Lagrange polynomial

$$p_1(x) = L_0(x)f_0 + L_1(x)f_1 = \frac{(x - x_1)}{(x_0 - x_1)} f_0 + \frac{(x - x_0)}{(x_1 - x_0)} f_1$$

Second degree interpolation 2.8.2

The second-degree polynomial $p_2(x)$ or quadratic interpolation is the interpolation of the points or nodes $(x_0, f_0), (x_1, f_1), (x_2, f_2)$ which by Lagrange's idea is

$p_2(x) = L_0(x)f_0 + L_1(x)f_1 + L_2(x)f_2$, where:

$$L_0(x) = \frac{l_0(x)}{l_0(x_0)} = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

$$L_1(x) = \frac{l_1(x)}{l_1(x_1)} = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

$$L_2(x) = \frac{l_2(x)}{l_2(x_2)} = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

Next, is the general Lagrange interpolation polynomial from Erwin Kreyszig (2008) on which the examples and our results are based.

Theorem 2.9

For general n we obtain

$$f(x) \approx p_n(x) = \sum_{k=0}^{n} L_k(x) f_k = \sum_{k=0}^{n} \frac{l_k(x)}{l_k(x_k)} f_k$$

Example 2.10 illustrates 2.8.2 and 2.9.

Example 2.10

If $f(1) = -3$, $f(3) = 9, f(4) = 30, f(6) = 132$, Find the Lagrange's interpolation polynomial of f(x).

Solution

From the given data we have.

X	1	3	4	6
F	-3	9	30	132

The interpolation points are $(x_0, f_0), (x_1, f_1), (x_2, f_2), (x_{3,}, f_3)$ which by 2.8 and 2.9, the Lagrange interpolation polynomial is:

$p_3(x) = L_0(x)f_0 + L_1(x)f_1 + L_2(x)f_2 + L_3(x)f_3$, where:

L0(x) $= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)}$

$= -\frac{1}{30}(x^3 - 13x^2 + 54x - 72)$

L1(x) $= \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x-1)(x-3)(x-6)}{3-1)(3-4)(3-6)}$

$= \frac{1}{6}(x^3 - 11x^2 + 34x - 24)$

L2(x) = $\frac{(x-x_0)(x-x_1)(z-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x-1)(x-3)(z-6)}{(4-1)(4-3)(4-6)}$

$= -\frac{1}{6}(x^3 - 10x^2 + 27x - 18)$

L3(x) $= \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)}$

$= \frac{1}{6}(x^3 - 8x^2 + 19x - 12)$

Substituting $L_j(x)$ and f_j, $j = 0, 1, 2, 3$ we get

P(x) = $[-\frac{1}{30}(x^3 - 13x^2 + 54x - 72)](-3) + [\frac{1}{6}(x^3 - 11x^2 + 34x - 24)](9) +$

$[-\frac{1}{6}(x^3 - 10x^2 + 27x - 18)](30) + [\frac{1}{6}(x^3 - 8x^2 + 19x - 12)](132)$

Which gives p(x) = $x^3 - 3x^2 + 5x - 6$

3 RESULTS

Remark 3.1

From the formular in theorem 1.18 we obtain the table for the minimum non - reducible representation min (G) and cardinality (Card (G)) of finite groups G whose cardinality is less than 100 where card (G) = p^n, p is an even prime with $2 < n < 7$, n is an integer and |G| = Card(G).

Card (G)	8	16	32	64
Min rep (G)	5	7	11	19

Table 3.1

From the formular in theorem 1.19 we obtain the table for the maximum non - reducible representation max (G) and cardinality (Card (G)) of finite groups G whose cardinality is less than 100 where card (G) = p^n , p is an even prime with $2 < n < 7$, n is an integer

Card (G)	8	16	32	64
Max rep (G)	5	10	20	40

Table 3.2

These tables provide the input data we used to obtain the next two results in this paper

It is to be noted that the values in the tables are not ordinary figures but values that carry the properties of finite groups which are abstract structures.

Our motivation for using the Lagrange interpolation is due to the fact that it permits the use of evenly and unevenly spaced values in deriving functions. Our data have these features

Result 3.2

Using Lagrange interpolation polynomial, the linear polynomial for estimating the minimum number of non-reducible rep for 2 - groups is $f(x) = \frac{1}{4}x + 3$

Solution

Let card(G) = x and min rep (G) = f(x). then from table 3.1 we have

X	8	16	32	64
f(x)	5	7	11	19

From the data the interpolation nodes are:

$$(x_0, f_0) = (8, 5): (x_1, f_1) = (16, 7): \left(x_{2,} f_2\right) = (32, 11): (x_3, f_3) = (64, 19)$$

From 2.8 and 2.9, the Lagrange's interpolating polynomial for the data is given by

$p_3(x) = L_0(x)f_0 + L_1(x)f_1 + L_2(x)f_2 + L_3(x)f_3$, where:

$$p_3(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f_0 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f_1 + \frac{(x-x_0)(x-x_1)(z-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f_3$$

$$p_3(x) = \frac{(x-16)(x-32)(x-64)}{(8-16)(8-32)(8-64)}(5) + \frac{(x-8)(x-32)(x-64)}{(16-8)(16-32)(16-64)}(7) + \frac{(x-8)(x-16)(z-64)}{(32-8)(32-16)(32-64)}(11) + \frac{(x-8)(x-16)(x-x_2)(x-32)}{(64-8)(64-16)(64-32)}(19)$$

$$p_3(x) = \frac{(x^2-48x+572)(x-64)}{(-8)(-24)(-56)}(5) + \frac{(x^2-40x+256)(x-64)}{(8-)(16)(-48)}(7) + \frac{(x^2-24x+128)(z-64)}{(24)(-16)(-32)}(11) + \frac{(x^2-24x+128)(x-32)}{(56)(48)(32)}(19)$$

$$p_3(x) = \frac{(x^3-112x^2+3584x-32768)}{-10752}(5) + \frac{(x^3-104x^2+2816x-1638)}{6144}(7)$$

$$+ \frac{(x^3-88x^2+1664x-8192)}{-12288}(11) + \frac{(x^3-56x^2+896x-4096)}{86016}(19)$$

$$p_3(x) = \frac{(5x^3-560x^2+17920x-163840\ )}{-10752} + \frac{(7x^3-728x^2+19712x-114688)}{6144}$$
$$+ \frac{(11x^3- 968x^2+18304x-90112)}{-12288} + \frac{(19x^3- 1064x^2+17024x-77824)}{86016}$$
$$p_3(x) = \left(-\frac{5}{10752} + \frac{7}{6144} - \frac{11}{12288} + \frac{19}{86016}\right)x^3 + \left(\frac{560}{10752} - \frac{728}{6144} + \frac{968}{12288} - \frac{1064}{86016}\right)x^2$$
$$+ \left(-\frac{17920}{10752} + \frac{19712}{6144} - \frac{18304}{12288} + \frac{17024}{86016}\right)x + \left(\frac{163840}{10752} - \frac{114688}{6144} + \frac{90112}{12288} - \frac{77824}{86016}\right)$$

Which simplifies to: $p_3(x) = f(x) = \frac{1}{4}x + 3$

Result 3.3

Using Lagrange's interpolation polynomial the linear polynomial for estimating he max numner on non reducible rep for 2 - groups is f(x) = 5x/8 $f(x) = \frac{5}{8}x$

Solution

Let card(G) = x and min rep (G) = f(x). then from table 3.2 we have

X	8	16	32	64
f(x)	5	10	20	40

From the data the interpolation nodes are:

$$(z_0, f_0) = (8, 5): (\ x_1, f_1) = (16, 10):\ (x_{2,}\ f_2\ ) = (32, 20\ ):\ (x_3, f_3) = (64, 40)$$

From 2.8 and 2.9, and 3.2 the Lagrange's interpolating polynomial for the data is given by $p_3(x) = \ L_0(x)f_0 + \ L_1(x)f_1 + \ L_2(x)f_2 + L_3(x)f_3$, where:

from table 3.2 we repeat the same process as in table 3.1. The x – values are the same as in the first table. So, we continue with the expansion as done above and substituting the values of f(x) as follows:

$$p_3(x) = \frac{(x^3-112x^2+3584x-32768\ )}{-10752}(5) + \frac{(x^3-104x^2+2816x-1638)}{6144}(10)$$
$$+ \frac{(x^3- 88x^2+1664x-8192)}{-12288}(20) + \frac{(x^3- 56x^2+896x-4096)}{86016}(40)$$
$$p_3(x) = \frac{(5x^3-560x^2+17920x-163840\ )}{-10752} + \frac{(10x^3-1040x^2+28160x-163840)}{6144}$$
$$+ \frac{(20x^3- 1760x^2+33280x-163840)}{-12288} + \frac{(40x^3- 2240x^2+28160x-163840)}{86016}$$
$$p_3(x) = \left(-\frac{5}{10752} + \frac{10}{6144} - \frac{20}{12288} + \frac{40}{86016}\right)x^3 + \left(\frac{560}{10752} - \frac{1040}{6144} + \frac{1760}{12288} - \frac{2240}{86016}\right)x^2$$
$$+ \left(-\frac{17920}{10752} + \frac{28160}{6144} - \frac{33280}{12288} + \frac{35840}{86016}\right)x + \left(\frac{163840}{10752} - \frac{163840}{6144} + \frac{163840}{12288} - \frac{163840}{86016}\right)$$

Simplifying this we obtain: $p_3(x) = f(x) = \ \frac{5}{8}x$

4 DISCUSSION

Comparing our polynomials though linear with that of we can see that data from abstract structure can be generated using numerical interpolations

So numerical analysis can be used to generate schemes from which the properties of some abstract algebraic structures can be derived. Our results agree with those of Jelten et al 2017 and 2019 as stated in theorems 1.18 and 1.19. The results are flexible and we can input more integral values in between to determine the structures of more groups. For instance, we can find values of f(x) for x equal to 24 or even 48. For the former we have f(x) = 9 in the case of the scheme derived in 3.1 and 15 for scheme derived in 3.2 We have 15 and 30 using the two schemes respectively.

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