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BAYESIAN VECTOR ZERO-INFLATED INGARCH MODEL: A CASE STUDY OF COVID-19 IN NIGERIA.

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Abstract

This study applied a Bayesian Vector Zero-Inflated Generalized Poisson INGARCH (ZIGP-INGARCH) model to captures the multivariate dynamics of COVID-19 time series data in Nigeria, characterized by overdispersion, excess zeros, and inter-variable dependence. The analysis utilizes daily counts of new cases, deaths, and recoveries from March 2020 to May 2023, obtained from the Nigeria Centre for Disease Control (NCDC). The model jointly estimated the conditional means of the three epidemiological indicators while accounting for both temporal and cross-sectional dependencies using a flexible INGARCH framework embedded with zero-inflated generalized Poisson marginals. Bayesian inference is performed using Hamiltonian Monte Carlo (HMC), ensuring robust parameter estimation and uncertainty quantification. Results reveal that the ZIGP-INGARCH(1,1) model captures significant moving average dynamics in recovery-to-case interactions ($B_{13} = 0.44$), mild overdispersion in new and recovered cases ($\phi_1 \approx 0.81$, $\phi_3 \approx 0.80$), and a moderate zero-inflation probability for death counts ($\pi_2 \approx 0.45$). Model adequacy was confirmed through residual diagnostics, AIC/BIC comparisons, and convergence metrics (R-hat $\approx$ 1.000). This research demonstrates the utility of Bayesian ZIGP-based multivariate models in understanding pandemic dynamics and guiding effective forecasting and policy response.

KEYWORDS: Bayesian inference, ZIGP-INGARCH, COVID-19, zero-inflation, multivariate time series

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INTRODUCTION

Time series models such as the Autoregressive (AR), Moving Average (MA), and Autoregressive Moving Average (ARMA) models are primarily developed for univariate, continuous, and stationary data. These models are aimed to capture autocorrelation within a single variable over time. However, many real-world phenomena particularly with the epidemiology data like the COVID 19 often exhibiting overdispersion and excess number of zeros (Zhu *et al*.,2020). To address such complexities, it requires a multivariate framework which allow for a simultaneous modeling of multiple time-dependent count series. These multivariate models not only capture dynamic temporal dependencies within each series but also account for cross-dependencies between them. Classical examples of models with such qualities are the Vector Autoregressive (VAR), Vector Autoregressive Moving Average (VARMA), and Multivariate GARCH (MGARCH) models (Lütkepohl, 2005; Heinen *et al*., 2007). However, such models are not directly suitable for discrete-valued, non-Gaussian, and zero-inflated data.

To model such data, the ideal of linking the Generalized Linear Models (GLMs) with the time series framework was introduced by Davis *et al.* 2001. Benjamin *et al*., 2003 leaverage on this development to model the Generalized Autoregressive Moving Average (GARMA**)**. the INGARCH (Integer-valued GARCH) models which emerged as a discrete analog to GARCH models, accommodating the conditional mean structure of count series through distributions like the Poisson or Negative Binomial and so on Davis *et al.* (2021). The foundational theory of INGARCH models stems from the GARMA-GARCH framework (Benjamin *et al*., 2003), initially restricted to exponential family distributions. Ferland *et al*. (2006) advanced this by introducing the INGARCH(p, q) model, inspired by Bollerslev *et al*. (1986), which uses a Poisson conditional distribution that evolves with past values, imitating GARCH dynamics. Subsequent work expanded this framework using alternative discrete distributions, such as the negative binomial (Zhu, 2011), the generalized Poisson (Zhu, 2012), and the generalized compound Poisson (Gonçalves *et al*., 2023).

While existing multivariate models in time series such as Multivariate Count Autoregression (Fokianos, 2020), multivariate INGARCH (Heinen *et al*., 2007), and state-space models (Shapovalova *et al*., 2022) have been widely applied, they struggle with count data that is overdispersed, and often zero-inflated (Zhang, 2022). The aim of this study is to apply COVID-19 data to Bayesian Vector Zero Inflated Generalized Poisson (INGARCH) models

MATERIALS AND METHODS

This study focuses on the Bayesian Vector Zero-Inflated Generalized Poisson (ZIGP-INGARCH)**.** The framework is built on Zero-inflated Generalized Poisson-INGARCH and It is restricted to daily COVID-19 data (new confirmed cases, Death case and Recovery) from National Centre for Disease Control in Nigeria, covering the period from March 2020 to May 2023, employing the Bayesian method for parameter estimation.

The INGARCH (Integer-valued GARCH) model extends the ARMA framework to count data by modeling the conditional mean $\lambda_t = E[Y_t \mid Y_{t-1}, Y_{t-2}, \dots]$ rather than the observations directly. Since count variables have non-integer expected values, this approach avoids negative predictions through appropriate parameter constraints. Despite the name suggesting an ARMA-like model for integer values, INGARCH mirrors the structure of GARCH models. For instance

GARCH(1,1):

$$\sigma_t^2 = \beta_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \quad (1)$$

INGARCH(1,1):

$$\lambda_t = \beta_0 + \alpha_1 Y_{t-1} + \beta_1 \lambda_{t-1} \quad (2)$$

Thus, the INGARCH model uses a similar recursive structure, but for means instead of variances.

The general INGARCH(p, q) model (Ferland *et al*., 2006) is defined as:

$$\lambda_t = d + \sum_{i=1}^{p} A_i Y_{t-i} + \sum_{j=1}^{q} B_j \lambda_{t-j} \quad (3)$$

where $\boldsymbol{Y_t}$ is a discrete-valued process (e.g., Poisson, NB, ZINB), $\boldsymbol{\lambda_t > 0}$, $\boldsymbol{d > 0}$ and $\boldsymbol{A_i, B_j \geq 0}$. If $\boldsymbol{q = 0}$ we refer to it as an INARCH(p) model.

The model is fully specified if the conditional distribution of $\boldsymbol{Y_t \mid \mathcal{F}_{t-1}}$ has been fixed. If $\boldsymbol{Y_t \mid \mathcal{F}_{t-1}}$ has Poisson distribution is called Poisson INGARCH (or linear Poisson autoregressive) model (Fokianos *et al*., 2009, Ferland *et al*., 2006) showed that the condition

$$\sum_{i=1}^{p} \boldsymbol{A_i} + \sum_{j=1}^{q} \boldsymbol{B_j} < \boldsymbol{0}$$

ensures a strictly stationary process with finite moments. The stationarity mean and variances given as

$\boldsymbol{\mu = \frac{\beta_0}{1-\sum_{i=1}^{p}\alpha_i - \sum_{j=1}^{q}\beta_j}}$. $\boldsymbol{Var[Y_t] = \mu + Var[\lambda_t]}$

indicating over dispersion within (GLM) framework, we can construct INGARCH models with nonlinear autocorrelation structures by introducing an additional link function. A link function transforms the conditional mean $\lambda_t = E[\,Y_t \mid \mathcal{F}_{t-1}\,]$ to the real line:

$g(\lambda_t) = \eta_t$

where η_t is a linear or nonlinear function of past values. The linear case is given in (3) where nonlinear is give as

$$\log \lambda_t = \beta_0 + \alpha_1 \log(Y_{t-1} + 1) + \beta_1 \log(\lambda_{t-1}). \quad (4)$$

Fokianos and Tjøstheim (2011):

$$\lambda_t = \exp(\beta_0) \cdot (Y_{t-1} + 1)^{\alpha_1} \cdot (\lambda_{t-1})^{\beta_1}$$

If α_1 and β_1 have the same sign, $|\alpha_1 + \beta_1| < 1$, Otherwise $\alpha_1^2 + \beta_1^2 < 1$, similarly, the

Multivariate extension of INGARCH (1,1) by Heinen *et al*.,(2003) is given as

$$\boldsymbol{\lambda_t = d_t + AY_{t-1} + B\lambda_{t-1}} \quad (5)$$

were **A**, **B** a $d \times d$ vector coefficient and d_t is a $d \times 1$ vector intercept

stationary mean is given as

$$E[\mathbf{Y}_t] = (I - \boldsymbol{A} - \boldsymbol{B})^{-1}\boldsymbol{\beta}_0 \quad (6)$$

Log-linear Multivariate INGARCH(1,1) (Fokianos*et al*., 2020), Cui *et al*., (2020)

$$\log(\boldsymbol{\lambda}_t) = \boldsymbol{d}_t + \boldsymbol{A}log(\boldsymbol{Y}_{t-1} + 1) + \boldsymbol{B}\log(\boldsymbol{\lambda}_{t-1}) \quad (7)$$

Model Specification

This section give specific account of the model used in this research work, we use used Zero-Inflated Generalized Poisson INGARCH model. Three COVID-19 variables were used (New Cases, Death Case and Recovery

Zero- Inflated Generalized Poisson INGARCH model

For a random variable Y_t is zero-inflated Genralized Poisson $(\boldsymbol{\lambda}_t, \theta)$ the pmf is given as

$$P(Y = k) = \begin{cases} \omega + (1-\omega).GP(0;\lambda,\theta)\,, if k = 0 \\ (1-\omega).GP(k;\lambda,\theta), \quad if\ k > 0 \end{cases} \quad (8)$$

Where $\lambda > 0$, $\theta \in (-1.1)$ and $0 < \omega < 1$

$$GP(k;\lambda,\theta) = \frac{\lambda(\lambda+\theta k)^{k-1}e^{-(\lambda+\theta k)}}{k!} \quad (9)$$

Mean and variance is given as $E[Y] = \{1-\pi\}\frac{\lambda}{1-\theta}$

$$Var[Y] = \{1-\pi\}\frac{\lambda}{(1-\theta)^3} + \pi\{1-\pi\}\frac{\lambda^2}{(1-\theta)^2}$$

Proof; the mean and variance of ZIGP is derived from the Generalized Poisson. The from (9) the mean is generated using the probability generated function (GPF)

$$G(s) = \sum_{k=0}^{\infty} s^x P(Y = k)$$

$G(s) = e^{\theta(u(s)-1\}}$ where $u(s)$ satisfies

$u(s) = \theta e^{\theta(u(s)-1\}}$

The mean is obtained by differentiating $G(s)$ at $s = 1$

$E[Y] = G^{'}(1)$

$u^{'}(s) = e^{\theta(u(s)-1\}} + s\theta u^{'}(s)e^{\theta(u(s)-1\}}$

At $s = 1$, $u'(1) = 1 + \theta u'(1)$

Hence $u'(1) = \frac{1}{1-\theta}$

Now differentiating $G(s)$:

$G^{'}(s) = \lambda u'(s)G(s)$

$G'(1) = \lambda u'(s)G(1) = \lambda\left(\frac{1}{1-\theta}\right) \times 1 = \frac{\lambda}{1-\theta}$

$E[Y] = \frac{\lambda}{1-\theta}$ (38)

Variance (Var(Y) Dérivation

The variance is obtained from the second factorial moment

$Var(X) = G''(s) + G'(s) - [G'(s)]^2$ (10)

To find $G''(1)$, we need to find $u''(s)$ from $u'(s)$

$$u''(s) = \theta u'(s)e^{\theta(u(s)-1\}} + \theta u'(s)e^{\theta(u(s)-1\}} + s\theta u''(s)e^{\theta(u(s)-1\}} + s\theta^2(u'(s))^2 e^{\theta(u(s)-1\}}$$

At $s = 1, u(1) = 1 \; and \; u'(1) = \frac{1}{1-\theta}$

$u''(1) = \theta u'(1) + \theta u'(1) + \theta u''(1) + \theta^2(u'(1))^2$

substitute$u'(1)$

$u''(1) = \frac{2\theta}{1-\theta} + \theta u''(1) + \frac{\theta^2}{(1-\theta)^2}$

$u''(1)(1-\theta) = \frac{2\theta}{1-\theta} + \frac{\theta^2}{(1-\theta)^2}$

$u''(1) = \frac{2\theta}{(1-\theta)^2} + \frac{\theta^2}{(1-\theta)^3}$

To compute $G''(1)$

$G''(1) = \lambda u''(1) + \lambda(u'(1))^2 = \lambda(\frac{2\theta}{(1-\theta)^2} + \frac{\theta^2}{(1-\theta)^3}) + \lambda(\frac{1}{1-\theta})$

$G''(1) = \frac{\lambda(2\theta+\theta^2)}{(1-\theta)^3} + \frac{\lambda}{(1-\theta)^2}$

Substitute $G'(1)$ and $G''(1)$ in (a)

$$Var(X) = \frac{\lambda(2\theta + \theta^2)}{(1-\theta)^3} + \frac{\lambda}{(1-\theta)^2} + \frac{\lambda}{1-\theta} + \frac{\lambda^2}{(1-\theta)^2}$$

$$= \frac{\lambda}{(1-\theta)^3}(2\theta + \theta^2 + (1-\theta) + (1-\theta)^2 - \lambda(1-\theta))$$

But since $E[Y] = \frac{\lambda}{1-\theta}$,

$$Var(X) = \frac{\lambda}{(1-\theta)^3} \quad (11)$$

Mean and variance for ZINB. Since ZIGP is a mixture distribution, its mean is

$$E[Y] = (1-\pi)E_{GP}[Y]$$

Substitute $E[Y] = \frac{\lambda}{1-\theta}$,

$$E[Y] = (1-\pi)\frac{\lambda}{1-\theta}$$

Variance of ZIGP

$$Var(Y) = (1-\pi)(Var_{GP}(Y) + \pi(E_{GP}[Y])^2$$

$$= (1-\pi)(\frac{\lambda}{(1-\theta)^3} + \pi(\frac{\lambda}{1-\theta})^2)$$

Hence

$$Var(Y) = (1-\pi)\ \frac{\lambda}{(1-\theta)^3} + \pi(1-\pi)\frac{\lambda^2}{(1-\theta)^2} \quad (12)$$

Hence (11) is the variance of GP Goncalves *et al.*, (2023).

Let $\mathbf{Y}_t = (Y_{1t}, Y_{2t}, \dots, Y_{dt})$ be vectors of count variables, $\boldsymbol{\lambda}_t = (\lambda_{1t}, \lambda_{2t}, \dots, \lambda_{dt})$ are conditional, $\boldsymbol{\theta} = (\theta_1, \dots, \theta_d)^T$ are dispersion parameters for generalized Poisson and $\boldsymbol{\pi}_t = (\pi_{1t}, \dots, \pi_{dt})$ are zero-inflated probalities. The conditional mean $\mathbf{Y}_t\ |\boldsymbol{\mathcal{F}}_{t-1}$ is given as

$$\begin{cases} \mathbf{Y}_t\ |\boldsymbol{\mathcal{F}}_{t-1} \sim ZIGP(\lambda, \theta, \pi) \\ \boldsymbol{\lambda}_t = \boldsymbol{d}_t + \boldsymbol{A}\boldsymbol{Y}_{t-1} + \boldsymbol{B}\boldsymbol{\lambda}_{t-1} \end{cases} \quad (13)$$

where **d** is a p-dimensional vector and **A**, **B** are d × d unknown matrices.. The elements of **d**, **A** and **B** are assumed to be positive such that $\boldsymbol{\lambda}_{i,t} > 0$and $0 < \pi < 1$. Given three variables Y_{1t}, Y_{2t}, and Y_{t2} as New Cases, Death Cases and Recovery respectivelyof COVID-19 variable the linear model is given as

$$\lambda_{i,t} = d_i + \sum_{j=1}^{3} A_{i,j} Y_{j,t-i} + \sum_{i=1}^{3} B_{i.j} \lambda_{j,t-i} \qquad i, j = 1,2,3 \qquad (14)$$

$$\lambda_{1,t} = d_1 + A_{11} Y_{1,t-1} + A_{12} Y_{2,t-1} + A_{13} Y_{3,t-1} + B_{11} \lambda_{1,t-1} + B_{12} \lambda_{2,t-1} + B_{13} \lambda_{3,t-1}$$

$$\lambda_{2,t} = d_2 + A_{21} Y_{1,t-1} + A_{22} Y_{2,t-1} + A_{23} Y_{3,t-1} + B_{21} \lambda_{1,t-1} + B_{22} \lambda_{2,t-1} + B_{23} \lambda_{3,t-1}$$

$$\lambda_{3,t} = d_3 + A_{31} Y_{1,t-1} + A_{32} Y_{2,t-1} + A_{33} Y_{3,t-1} + B_{31} \lambda_{1,t-1} + B_{32} \lambda_{2,t-1} + B_{33} \lambda_{3,t-1}$$

Matrix form is given as;

$$\begin{pmatrix} \lambda_{1,t} \\ \lambda_{2,t} \\ \lambda_{3,t} \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} + \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \\ Y_{3,t-1} \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} \begin{pmatrix} \lambda_{1,t-1} \\ \lambda_{2,t-1} \\ \lambda_{3,t-1} \end{pmatrix} \qquad (15)$$

where **d>0** is a p-dimensional vector and **A**, **B** are d × d unknown matrices. The elements of **d**, **A** and **B** are assumed to be positive such that $\boldsymbol{\lambda}_{i,t} > 0$ for all i and t. This also fellow a VARMA (p,q) such that Where $(\mathbf{A}_{i,})_{j=1}^{q} \geq 0$and $(\mathbf{B}_{i})_{i=1}^{q} \geq 0$ are d × d-dimensional matrices and all the element of $\boldsymbol{\lambda}_{i,t} > 0(\mathbf{A}_{i,})_{j=1}^{q}$ are the VAR coefficient, representing the autoregressive effects across multiple series Heinen *et al.* (2003) and Fokianos *et al.* (2021)

log-linear Model given as

$\boldsymbol{Y_{i,t}} \mid \boldsymbol{\mathcal{F}_{t-1}} \sim \boldsymbol{ZIGP}(\boldsymbol{r_i}, \boldsymbol{\lambda_{i,t}})$,i = 1,2,3,

$$\boldsymbol{log}\ (\boldsymbol{\lambda_t}) = \boldsymbol{d_t} + \boldsymbol{Alog}(\boldsymbol{Y_{t-1}} + \mathbf{1}) + \boldsymbol{Blog}(\boldsymbol{\lambda_{t-1}}) \qquad (15)$$

$$v_{i,t} = d_i + \sum_{j=1}^{3} A_{i,j} \log(Y_{j,t-i} + 1) + \sum_{i=1}^{3} B_{i.j} v_{j,t-i} \qquad i, j = 1,2,3$$

where $\boldsymbol{v}_t = log\boldsymbol{\lambda}_t$ is defined component wise (i.e.$\boldsymbol{v}_{i,t} = log\ \boldsymbol{\lambda}_{i,t}$) and 1_p denotes the p-dimensional vector which consists of ones (Tsamtsakir,.2023). the assumption for positivity **d**, **A** and **B** need not be for the log linear model except for $\boldsymbol{\lambda}_{i,t} > 0$

$$\log(\lambda_{1,t}) = d_1 + A_{11} \log(Y_{1,t-1} + 1) + A_{12} \log(Y_{2,t-1} + 1) + A_{13} \log(Y_{3,t-1} + 1) + B_{11} \log(\lambda_{1,t-1}) + B_{12} \log(\lambda_{2,t-1}) + B_{13} \log(\lambda_{3,t-1})$$

$$\log(\lambda_{2,t}) = d_2 + A_{21} \log(Y_{1,t-1} + 1) + A_{22} \log(Y_{2,t-1} + 1) + A_{23} \log(Y_{3,t-1} + 1) + B_{21} \log(\lambda_{1,t-1}) + B_{22} \log(\lambda_{2,t-1}) + B_{23} \log(\lambda_{3,t-1})$$

$$\log(\lambda_{3,t}) = d_3 + A_{31} \log(Y_{1,t-1} + 1) + A_{32} \log(Y_{2,t-1} + 1) + A_{33} \log(Y_{3,t-1} + 1) + B_{31} \log(\lambda_{1,t-1}) + B_{32} \log(\lambda_{2,t-1}) + B_{33} \log(\lambda_{3,t-1})$$

therefore the log-linear Vector ZIGP model (Fokianos *et.al.*, 2020) in matrix form is given as;

$$\begin{pmatrix}\log(\lambda_{1,t})\\ log(\lambda_{2,t})\\ \log(\lambda_{3,t})\end{pmatrix}=\begin{pmatrix}d_1\\ d_2\\ d_3\end{pmatrix}+\begin{pmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix}\begin{pmatrix}\log(Y_{1,t-1}+1)\\ \log(Y_{2,t-1}+1)\\ \log(Y_{3,t-1}+1)\end{pmatrix}+\begin{pmatrix}B_{11} & B_{12} & B_{13}\\ B_{21} & B_{22} & B_{23}\\ B_{31} & B_{32} & B_{33}\end{pmatrix}\begin{pmatrix}\log(\lambda_{1,t-1})\\ \log(\lambda_{2,t-1})\\ \log(\lambda_{3,t-1})\end{pmatrix}$$

ANALYSIS

Parameter to be estimated includes

$\theta - clayton\ parameter,\ d_t - intercept\ foe\ conditional\ mean,\ A - AR\ vector\ coefficient,$
$B - Mean\ paameter\ vector\ coefficient r_i - dispersion\ parameter \pi_{,t}$- parameter for excess zero

Prior distribution for the parameter to be estimated $r_i \sim Gama\ (2,1) == (\frac{b^a}{\Gamma(a)x^{a-1}})e^{-bx}$

$\pi \sim Beta\ (2,2) = \frac{1}{B(a,b)}x^{a-1}1-x^{b-1}.\quad A_i \sim halfN(0,0.5) = \frac{1}{\sigma\sqrt{2\pi}}e^{(\frac{-x^2}{2\sigma^2})}$

$B_i \sim halfN(0,0.5) = \frac{1}{\sigma\sqrt{2\pi}}e^{(\frac{-x^2}{2\sigma^2})},\qquad d_i \sim halfN(0,0.5) = \frac{1}{\sigma\sqrt{2\pi}}e^{(\frac{-x^2}{2\sigma^2})}$

The Bayesian estimation using Monte Carlos Markov Chain (MCMC) NUTS

The joint linear likelihood for all time points $t = 1, \dots, T$ and series$i = 1,2,3$ is:

$$L(Y_{it}/\theta) = \prod_{t=1}^{T}\prod_{i=1}^{3} p(Y_{it}/\lambda_{it}(\theta), r_i, \pi_i) = \prod_{t=1}^{T}\prod_{i=1}^{3}[\pi_i^{I(Y_{it}=0)}.(1-\pi)^{I(Y_{it}>0)}.GP(Y_{it}/\lambda_{it}, r_i)$$

Hence the linear posterior gievn as

$$P(\theta/y)\ \alpha\ L(\theta/y_{kt}) \times P(A_k) \times P(B_k) \times P(\phi_k) \times P(\pi_k) \times P(d_k)$$

Where $\theta = (A, B, d, \phi, \pi)$

Log-likelihood for MCMC

$$Log\ L = \sum_{t,i}[\mathbb{I}(Y_{it}=0)\log(\pi_i + (1-\pi_i)GP(0) + \mathbb{I}(Y_{it}>0)\ Log(1-\pi_i) + \log GP(Y_{it}))]$$

For $Y_{it} = 0,\qquad Log(\pi + (1-\pi)e^{-\lambda})$

For $Y_{it} > 0,\qquad \log(1-\pi) + log\lambda_t + (y_t - 1)\log(1+\phi y_t) - \lambda_t(1+\phi y_t) - \log(y_t!)$

Log priors

Log prior for $(A, B, d) \sim N(0,0.5)$

$\log P\ (A, B, d) = -\frac{1}{2}Log(2\pi\ .0.25) - \frac{(d-0)^2}{2\ .0.25}.\ A, B \geq 0\ and\ \ d\ \in \mathbb{R}\ hence\ d\ retain\ normal\ prior$

For $\phi \sim Gamma(1,1)$, Log P$(\phi) = -\phi + 0.log\phi$

For π Beta(1,1), Log P$(\pi) = 0$

Log Posterior Distribution

$$\text{Log}P(\theta/y) = logL(\theta/y_{kt}) + logP(A_k) + logP(B_k) + logP(\phi_k) + logP(\pi_k) + logP(d_k) + constant$$

$$LogP(\theta/y) = \sum_{t=1}^{T}\sum_{k=1}^{3} LogP(Y_{kt} = y_{kt}/\lambda_{kt}, \phi_k, \pi_k) + \sum_{k,i,j=1}^{3} [Logp(A_{ki}) + Log(B_{ki}) + logP(d_k)] + \sum_{k=1}^{3} [Log\ p(\phi_k) + logP(\pi_k)] + C$$

$$LogP(\theta/y) = \sum_{t=1}^{T}\sum_{k=1}^{3} \begin{cases} Log(\pi_k + (1-\pi_k)e^{-\lambda_{kt}} & if\ y_{kt} = 0 \\ \log(1-\pi_k) + log\lambda_{kt} + (y_{kt}-1)\log(1+\phi_k y_t) - \lambda_{kt}(1+\phi_k y_{kt}) - \log(y_{kt}!), & if\ y_{kt} > 0 \end{cases}$$
$$+ \sum_{k,i,j=1}^{3} \left[\frac{A_{ki}^2}{2.0.25} - \log(0.5\sqrt{2\pi})\right] \mathbb{I}_{A_{ki}>0} + \sum_{k,i,j=1}^{3} \left[\frac{B_{ki}^2}{2.0.25} - \log(0.5\sqrt{2\pi})\right] \mathbb{I}_{B_{ki}>0}$$
$$+ \sum_{k,i,j=1}^{3} \left[\frac{d_{ki}^2}{2.0.25} - \log(0.5\sqrt{2\pi})\right] + \sum_{k=1}^{3} [-\phi_k + 0.Log\phi_k - Log\Gamma(1)] + \sum_{k=1}^{3} [0] + C$$

Posterior Sampling via MCMC

We use the No-U-Turn Sampler (NUTS) to draw samples from the posterior:

Step 1: Initialize parameters $\theta^{(0)} = (\alpha^{(0)}, \beta^{(0)}, d^{(0)}, \phi^{(0)}, \pi^{(0)})$

Step 2: For each MCMC iteration $\boldsymbol{s} = 1, \ldots, S$

Step 3: Propose new parameters θ^* using Hamiltonian dynamics:

Step 4: Simulate trajectories in parameter space, guided by gradients of $logP(\theta \mid y)$.

Step 5: Accept/Reject θ^* based on the Metropolis-Hastings ratio:

$$r = \frac{p(\theta^* \mid y)}{p(\theta^{(s-1)} \mid y)}$$

Step 6: Store $\theta^{(s)} = \theta^*$ if accepted, else $\theta^{(s)} = \theta^{(s-1)}$

Output: Posterior samples $\{\theta^{(1)}, \ldots, \theta^{(s)}\}$.

Gradient Calculation:

Compute $\nabla\theta log\, p(\theta \mid y)$ for NUTS using automatic differentiation (JAX in Python).

Example, gradient for any parameter α in θ:

$$\frac{\partial log\, p(\theta \mid y)}{\partial y_{kt}} = \sum_{t=0}^{T} \frac{\partial log\, P(Y_t = y_t/\theta)}{\partial \lambda_t} \cdot \frac{\partial \lambda_t}{\partial y_{kt}}$$

1. Conditional Mean ($\boldsymbol{\lambda}_t$) Gradient:

$$\frac{\partial \boldsymbol{\lambda}_t}{\partial A} = \frac{\boldsymbol{\lambda}_t \cdot log(1 + y_{t-1})}{1 + Alog(1 + y_{t-1}) + Blog(\lambda_{t-1})}.$$

MCMC (e.g., Random Walk Metropolis) suffers in high dimensions but HMC uses gradient information to propose more efficient moves

HMC Algorithm (Hoffman & Gelman, 2014):

Augment with momentum$\boldsymbol{r} \sim N(0, \boldsymbol{M})$

Hamiltonian dynamics is given as

$H(\theta, r) = -\log(\theta/\boldsymbol{Y}) + \frac{1}{2} r^T M^{-1} r$

Leapfrog integrator (discretized dynamics)

$\boldsymbol{r}^{(t+\epsilon/2)} = \boldsymbol{r}^{(t)} + \frac{\epsilon}{2} \nabla_\theta \log(\theta^{(t)}/\boldsymbol{Y})$

$\theta^{(t+\epsilon)} = \theta^{(t)} + \epsilon \boldsymbol{M}^{-1} r^{(t-\epsilon/2)}$

$\boldsymbol{r}^{(t+\epsilon)} = \boldsymbol{r}^{(t+\epsilon/2)} + \frac{\epsilon}{2} \nabla_\theta \log(\theta^{t+\epsilon}/\boldsymbol{Y})$

Metropolis acceptance:

$\alpha = \min(1, \exp(\theta^{(t)}, \boldsymbol{r}^{(t)} - H(\theta^*, \boldsymbol{r}^*)))$

RESULTS

Table 1: Descriptive Statistics of COVID-19 Variables

Variable	Count	Mean	Std Dev	Min	Median	Max	Skewness	Kurtosis
NEW CASES	1187	229.09	363.47	0.0	79.0	4006.0	3.113	15.207
DEATH CASES	1187	2.616	5.589	0.0	0.0	93.0	7.259	95.104
RECOVERY	1186	201.99	459.79	0.0	48.5	8228.0	7.774	98.262

In Table 1, the descriptive statistics shows that the Mean = 229.09, Std Dev = 363.47 for new case as such Overdispersion is present (variance > mean) which implies Poisson inadequate. For the death

cases, similar in the case of death cases of Mean = 2.616, Std Dev = 5.589 Standard Poisson or NB models likely misspecify this structure, Sinilarly Mean = 201.99, Std Dev = 459.79 Standard count models would not fit well due to extreme dispersion (Serhiyenko, 2015, Cui et al 2018) .

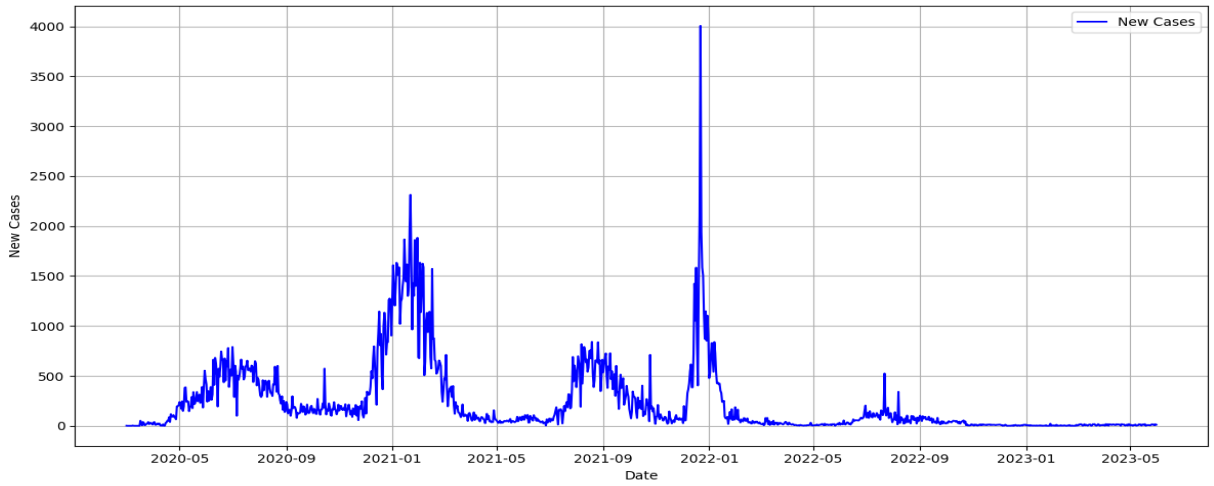


Figure 1: Plot of New Cases of COVID-19 2020 – 2023 from 2020 - 2023

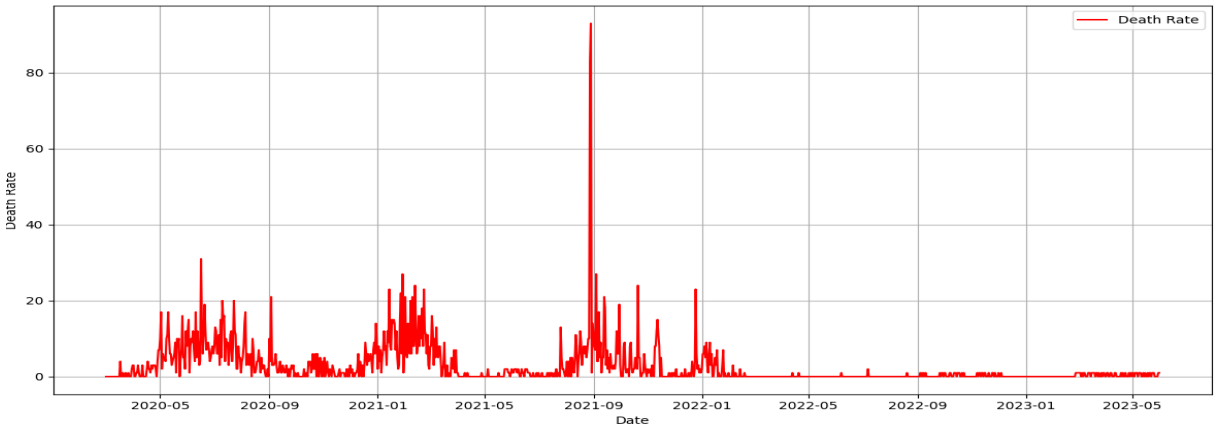


Figure 2: Time Plot of Death Cases of COVID-19 From 2020 – 2023 from 2020 - 2023

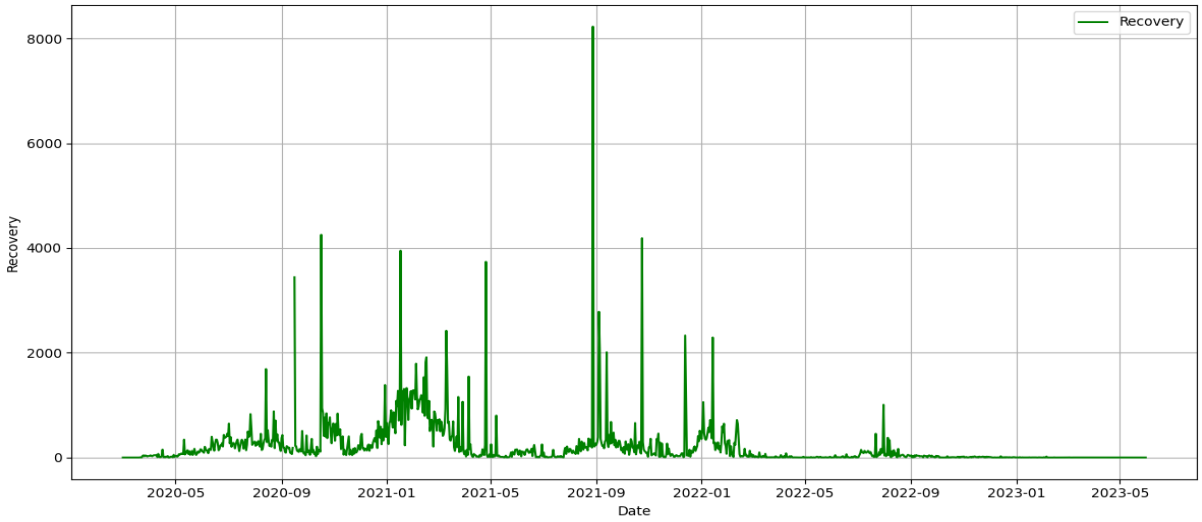
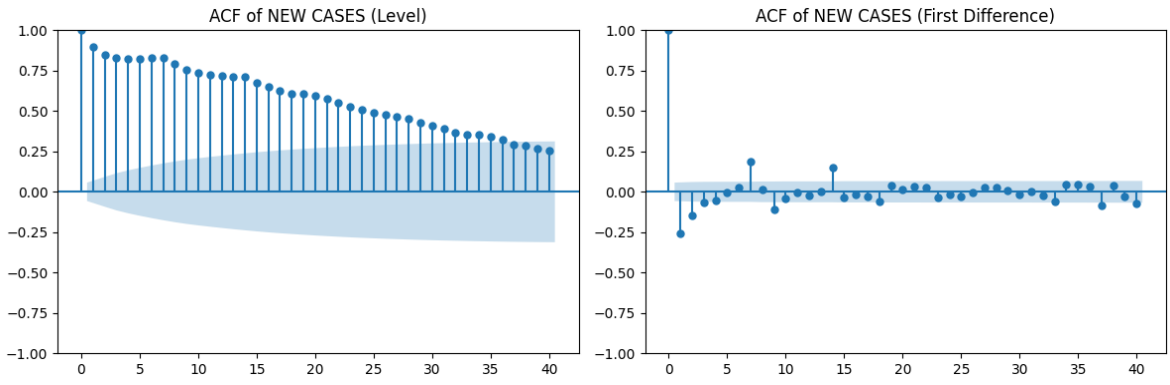


Figure 3: Time Plot for Recovery of COVID-19 from 2020 - 2023

The time series plots indicate multiple waves of the fro all the three variables, with the largest peak in early 2022, while earlier waves in 2020 and 2021 suggest initial outbreaks and persistent increase until log down measures . The data clearly shows overdispersion, supporting the use of Negative Binomial models instead of Poisson. Death cases increased gradually at first, peaked significantly from late 2020 to late 2021, and then declined, Recovery trends show steady growth, with a peak of 8,000 recoveries over the three years.



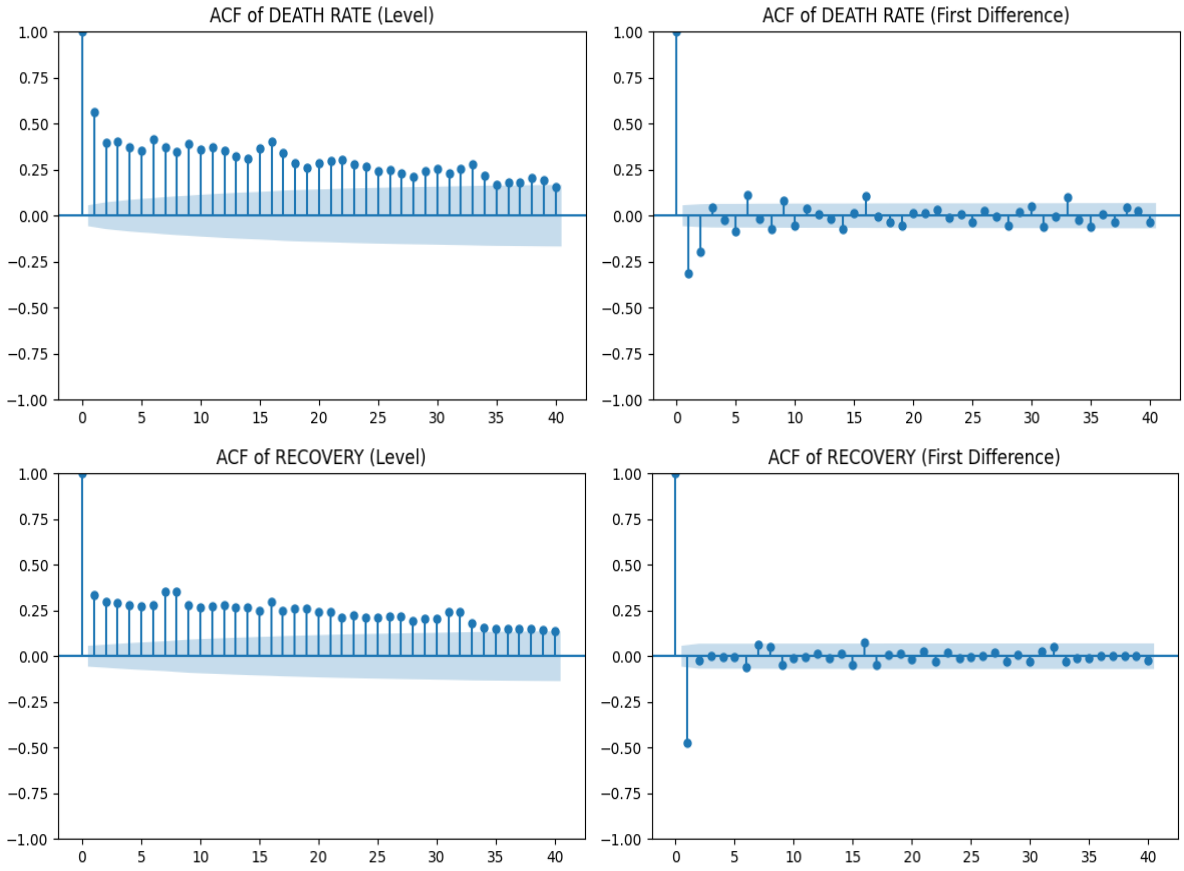


Figure 4: Autocorrelation plot for COVID-19 Data at level and first difference

Figure 4 shows all variables at Level ACF decay Moderate to strong autocorrelation up to lag 40 implying non-stationary and First Difference ACF for all variables Mostly within confidence bands achieving stationarity.

Table 2. Posterior Estimates for Vector Linear ZIGP INGARCH Model

Parameter	Mean	SD	HDI (3%, 95%)
A[1,1]	0.101	0.095	[0.000, 0.281]
A[1,2]	0.107	0.096	[0.000, 0.284]
A[1,3]	0.206	0.131	[0.001, 0.439]
A[2,1]	0.094	0.085	[0.000, 0.254]
A[2,2]	0.095	0.086	[0.000, 0.254]
A[2,3]	0.104	0.106	[0.000, 0.307]
A[3,1]	0.185	0.115	[0.000, 0.384]
A[3,2]	0.111	0.097	[0.000, 0.288]

Parameter	Mean	SD	HDI (3%, 95%)
A[3,3]	0.133	0.114	[0.000, 0.350]
B[1,1]	0.209	0.156	[0.000, 0.490]
B[1,2]	0.163	0.127	[0.000, 0.400]
B[1,3]	0.218	0.140	[0.000, 0.462]
B[2,1]	0.142	0.108	[0.000, 0.334]
B[2,2]	0.122	0.107	[0.000, 0.323]
B[2,3]	0.137	0.106	[0.000, 0.330]
B[3,1]	0.220	0.140	[0.001, 0.466]
B[3,2]	0.166	0.129	[0.000, 0.402]
B[3,3]	0.245	0.168	[0.000, 0.537]
d_1	0.035	0.312	[–0.557, 0.671]
d_2	0.041	0.329	[–0.625, 0.674]
d_3	–0.084	0.325	[–0.724, 0.525]
φ_1	0.958	0.022	[0.917, 0.999]
φ_2	0.425	0.097	[0.250, 0.606]
φ_3	0.938	0.027	[0.889, 0.989]
$\pi[1]$	0.130	0.079	[0.001, 0.265]
$\pi[2]$	0.457	0.082	[0.298, 0.599]
$\pi[3]$	0.171	0.082	[0.018, 0.316]

HDI (Highest Density Interval), R-hat (R̂) (Gelman-Rubin statistic)

1. Estimated Parameters

$$\phi = [0.958, 0.425, 0.938],\ \pi = [0.130, 0.457, 0.171]$$

Matrix form is given as;

$$\begin{pmatrix}\lambda_{1,t}\\ \lambda_{2,t}\\ \lambda_{3,t}\end{pmatrix} = \begin{pmatrix}0.035\\ 0.036\\ -0.087\end{pmatrix} + \begin{pmatrix}0.103 & 0.106 & 0.209\\ 0.092 & 0.094 & 0.103\\ 0.185 & 0.111 & 0.133\end{pmatrix}\begin{pmatrix}Y_{1,t-1}\\ Y_{2,t-1}\\ Y_{3,t-1}\end{pmatrix} + \begin{pmatrix}0.211 & 0.161 & 0.219\\ 0.142 & 0.123 & 0.137\\ 0.221 & 0.166 & 0.247\end{pmatrix}\begin{pmatrix}\lambda_{1,t-1}\\ \lambda_{2,t-1}\\ \lambda_{3,t-1}\end{pmatrix}$$

Conditional Mean Equations

$$\lambda_{1,t} = 0.035 + 0.101\,Y_{1,t-1} + 0.107\,2Y_{2,t-1} + 0.206\,Y_{3,t-1} + 0.209\,\lambda_{1,t-1} + 0.163\,\lambda_{2,t-1} + 0.218\,\lambda_{3,t-1}$$

$$\lambda_{2,t} = 0.041 + 0.094Y_{1,t-1} + 0.095\,Y_{2,t-1} + 0.104\,Y_{3,t-1} + 0.142\lambda_{1,t-1} + 0.122\,\lambda_{2,t-1} + 0.137\lambda_{3,t-1}$$

$$\lambda_{3,t} = -0.084 + 0.185\,Y_{1,t-1} + 0.111\,Y_{2,t-1} + 0.133\,Y_{3,t-1} + 0.220\,\lambda_{1,t-1} + 0.166\,\lambda_{2,t-1} + 0.245\lambda_{3,t-1}$$

The posterior estimates from the Vector Linear ZIGP INGARCH model in table 2 shows that most autoregressive (A) and moving average (B) coefficients are positive in agreement with), Goncalves *et al.* (2016) and Tsamtsakiri, (2023) indicating moderate dynamic interactions among new cases, deaths, and recoveries. The intercept terms (d_1=0.035, d_2=0.036, d_3=−0.084) show wide credible intervals including zero, implying limited standalone predictive power. The dispersion parameters (φ_1, φ_2, φ_3) indicates strong temporal dependence in new cases and recoveries, especially φ_1 (0.958) and φ_3 (0.938), while φ_2 (0.425) suggests more moderate persistence for deaths. The zero-inflation probabilities π indicate the death count process has the highest likelihood of excess zeros ($\pi[2] = 0.457$), consistent with the data's sparsity. In summary, the estimates support the suitability of the ZIGP INGARCH structure for modeling overdispersed, zero-inflated multivariate COVID-19 count data with temporal dependencies.

Table 3: Residual Diagnostics for Vector Linear ZIGP INGARCH

Series	Mean	Std Dev	Shapiro-Wilk (p)	K-S Test (p)	Ljung-Box (p, lag=10)
NEW CASES	0.41	0.15	0.0000	0.0001	0.1492
DEATH CASES	0.86	7.55	0.0000	0.0000	0.0000
RECOVERY	0.19	0.94	0.0000	0.0000	0.3244

Table 3 shows that all three series have non-normal residuals, supporting the use of ZIGP models. The Ljung-Box test suggests no significant autocorrelation in the residuals for New Cases (p = 0.1492) and Recovery (p = 0.3244), While residuals for New Cases and Recovery show no significant autocorrelation, Death Cases exhibit strong autocorrelation, suggesting the model may not fully capture its dynamics.

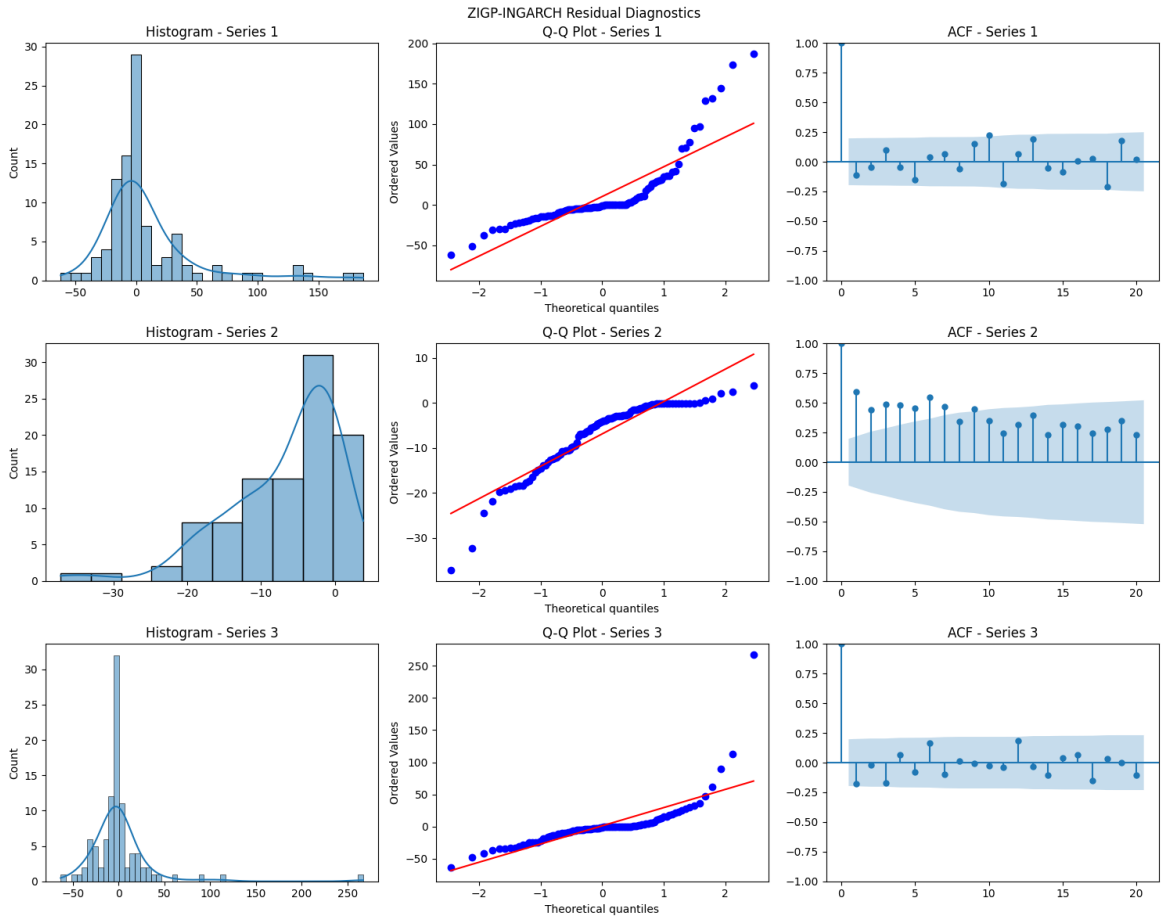


Figure 5: Residual Analysis for linear Vector ZIGP INGARCH

Table 4. Posterior Estimates for Log- linear Vector ZIGP INGARCH Model

Parameter	Mean	SD	HDI 3%	HDI 97%	R-hat
A[1,1]	0.381	0.076	0.237	0.522	1.000
A[1,2]	0.040	0.036	0.000	0.108	1.001
A[1,3]	0.016	0.015	0.000	0.044	1.001
A[2,1]	0.391	0.133	0.122	0.626	1.001
A[2,2]	0.242	0.168	0.001	0.538	1.000
A[2,3]	0.102	0.073	0.000	0.231	1.000
A[3,1]	0.023	0.020	0.000	0.058	1.000
A[3,2]	0.026	0.023	0.000	0.067	1.000
A[3,3]	0.118	0.052	0.025	0.214	1.001

B[1,1]	0.388	0.144	0.084	0.619	1.000
B[1,2]	0.027	0.028	0.000	0.076	1.001
B[1,3]	0.128	0.131	0.000	0.401	1.000
B[2,1]	0.065	0.057	0.000	0.169	1.001
B[2,2]	0.369	0.164	0.057	0.660	1.002
B[2,3]	0.056	0.051	0.000	0.150	1.002
B[3,1]	0.082	0.059	0.000	0.185	1.000
B[3,2]	0.028	0.025	0.000	0.072	1.000
B[3,3]	0.620	0.095	0.442	0.796	1.001
d[1]	3.021	0.557	1.971	4.006	1.000
d[2]	-2.384	0.610	-3.517	-1.255	1.001
d[3]	2.106	0.501	1.210	3.047	1.001
φ[1]	0.815	0.019	0.780	0.849	1.001
φ[2]	0.109	0.029	0.058	0.165	1.000
φ[3]	0.797	0.023	0.753	0.839	1.000
π[1]	0.024	0.021	0.000	0.062	1.000
π[2]	0.068	0.041	0.000	0.139	1.001
π[3]	0.055	0.033	0.000	0.113	1.001

HDI (Highest Density Interval), R-hat (R̂) (Gelman-Rubin statistic)

2. Estimated Parameters

$\phi[0.815\,, 0.109\ \,, 0.797], \pi = [0.024\,, 0068, 0.055]$

$$\begin{pmatrix} \log(\lambda_{1,t}) \\ log(\lambda_{2,t}) \\ \log(\lambda_{3,t}) \end{pmatrix} = \begin{pmatrix} 3.021 \\ -2.384 \\ 2.106 \end{pmatrix} + \begin{pmatrix} 0.381 & 0.040 & 0.040 \\ 0.391 & 0.242 & 0.102 \\ 0.023 & 0.026 & 0.118 \end{pmatrix} \begin{pmatrix} \log(Y_{1,t-1}+1) \\ \log(Y_{2,t-1}+1) \\ \log(Y_{3,t-1}+1) \end{pmatrix} + \begin{pmatrix} 0.388 & 0.027 & 0.128 \\ 0.065 & 0.396 & 0.056 \\ 0.082 & 0.028 & 0.620 \end{pmatrix} \begin{pmatrix} \log(\lambda_{1,t-1}) \\ \log(\lambda_{2,t-1}) \\ \log(\lambda_{3,t-1}) \end{pmatrix}$$

Conditional Mean Equations (log-link):

$$\log(\lambda_{1,t}) = 3.021 + 0.381 log(Y_{1,t-1}+1) + 0.040 log(Y_{2,,t-1}+1) + +0.016 \log(Y_{3,t-1}+1) + 0.388 \log \lambda_{1,t-1} + 0.027 \log(\lambda_{2,t-1}) + 0.128 \log(\lambda_{3,t-1})$$

$$\log(\lambda_{2,t}) = -2.384 + 0.391 \log(Y_{1,t-1}+1) + 0.242 \log(Y_{2,t-1}+1) + 0.102 \log(Y_{3,t-1}+1) + 0.065 \log(\lambda_{1,t-1}) + 0.369 \log(\lambda_{2,t-1}) + 0.056 \log(\lambda_{3,t-1})$$

$$log(\lambda_{3,t}) = 2.106 + 0.023 log(Y_{1,t-1}+1) + 0.026 log(Y_{2,t-1}+1) + 0.118 log(Y_{3,t-1}+1) + 0.082 log(\lambda_{1,t-1}) + 0.028 log(\lambda_{2,t-1}) + 0.620 log(\lambda_{3,t-1})$$

Table 18: Residual Diagnostics for Vector Log- Linear ZIGP INGARCH

Series	Mean	Std Dev	% |resid| > 1.96)	Normality (Q-Q)	ACF Decay
NEW CASES	0.668	0.692	5.05%	Normal	Acceptable
DEATH CASES	0.630	0.509	3.03%	Close to Normal	Acceptable
RECOVERY	0.632	0.632	3.03%	Slight Tail Risk	Acceptable

Series	Mean	Std Dev	% |resid| > 1.96)	Normality (Q-Q)	ACF Decay
NEW CASES	0.668	0.692	5.05%	Normal	Acceptable
DEATH CASES	0.630	0.509	3.03%	Close to Normal	Acceptable
RECOVERY	0.632	0.632	3.03%	Slight Tail Risk	Acceptable

In Table 17, the log-linear ZIGP-INGARCH model shows cross relationship with the variables, in New Cases, with A[1,1]= 0.381 and B[1,1]= 0.388 both having narrow HDIs that exclude zero. Death Cases is clearly driven by its past values shows A[2,1]=0.391, B[2,2]=0.369, while Recovery is dominated by B[3,3]=0.620, with a high HDI of [0.442, 0.796]. Significant dispersion is captured by ϕ1=0.815 and ϕ3=0.797, while ϕ2=0.109 reflects lower overdispersion in Death

Cases. Zero-inflation is minimal across series, with π values below 0.07. All R-hat values are approximately 1.000, confirming convergence. These results indicate strong self-dependence in New Cases and Recovery and limited cross-series influence while Death Rate shows weaker dynamics and lower overdispersion, all chains converged, supporting reliable inference.

These findings are in agreement with the works of Harvey *et al*., (2021) who observed strong autoregressive patterns but less predictable dynamics in mortality, likely due to underreporting in COVID-19 cases. Similarly, Albarracín *et al*. (2019) reported moderate to strong moving average (MA) effects dominating over AR effects in pandemic surveillance data, which supports the prominence of B terms in the present analysis.

Residual diagnostics in Table 4 show that mean residuals are moderately above zero, suggesting a possible slight bias in location. Standard deviations are all below 1, indicating no explosive behavior. Normality and ACF is acceptable, based on earlier Q-Q and ACF plots. In lines with Gelman *et al*. **(**2013), who emphasized the importance of R-hat diagnostics and effective sample sizes (ESS) in confirming posterior reliability, all of which were met here.

In conclusion, the linear ZIGP-INGARCH model underestimate autoregressive effects and shows weaker significance across key parameters, particularly in the A matrix. This difference suggests that the log-linear captured the multiplicative and exponential nature of epidemic dynamics in counts data more proportionally than additively over time.Hence, the log-linear ZIGP-INGARCH model provides a more robust and meaningful structure for modeling the dynamics of COVID-19-related counts data. It offers stronger autoregressive signals, improved estimation of overdispersion, and better identification of zero inflation, making it preferable for capturing the complex features in the data. Convergence diagnostics (R-hat $\approx$ 1.00) further support the reliability of these Bayesian estimates.

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